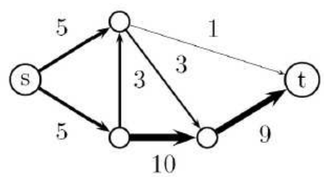
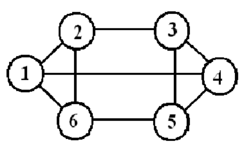
1> *Graph Terminology*

a) Network:

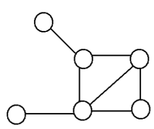
A network is a directed graph $N = (V, E)$ with a source node s (with $d_{out}(s) > 0$) and a terminal node t (with $d_{in}(t) > 0$). Moreover each edge has a strictly positive capacity $c(e) > 0$.



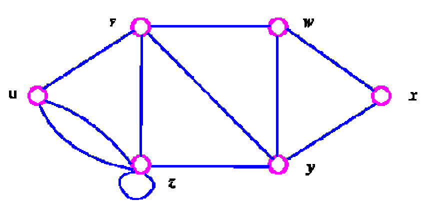
b) Regular Graph: A graph G is said to be **regular**, if every vertex of G has the same degree. If this degree is equal to r, then G is r-**regular** or **regular of degree** r.



c)Simple Graph: **simple graph**: an undirected graph without loop or multiple edges

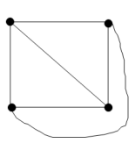


d) Trail: A walk in which no edge appears more than once



vzzywxy is a trail.

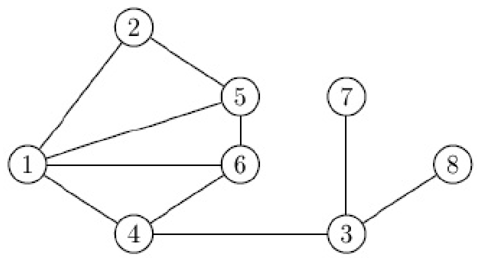
e) Planar Graph: A graph (or multigraph) *G* is called *planar* if *G* can be drawn in the plane with its edges intersecting only at vertices of *G*. Such a drawing of *G* is called an *embedding* of *G* in the plane.e.g.:



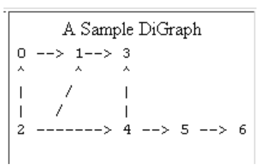
f) Adjacency list: an **adjacency list** representation of a graph is a collection of unordered lists of adjacent vertices, one for each vertex in the graph.

<u>For undirected graph:</u>

$$V(1) = \{2, 4, 5, 6\}$$
$$V(2) = \{1, 5\}$$
$$V(3) = \{4, 7, 8\}$$
$$V(4) = \{1, 3, 6\}$$
$$V(5) = \{1, 2, 6\}$$
$$V(6) = \{1, 4, 5\}$$
$$V(7) = \{3\}$$
$$V(8) = \{3\}$$



<u>For Directed graph:</u>



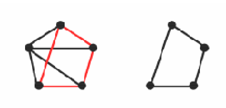
Digraph	
[0]	-> 1
[1]	-> 3
[2]	-> 0 -> 1 -> 4
[3]	NULL
[4]	-> 3 -> 5
[5]	-> 6
[6]	NULL

g) Adjacent vertices:

When two vertices u, v in $V(G)$ are endpoints of an edge, we say u and v are **adjacent**.

h)Subgraph:

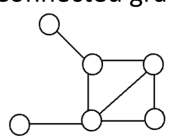
A **subgraph** H of a graph G, is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ satisfying the property that for every $e \in E(H)$, where e has endpoints $u, v \in V(G)$ in the graph G, then $u, v \in V(H)$ and e has endpoints u, v in H, i.e. the edge relation in H is the same as in G.



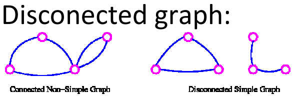
i) Connected Graph, Disconnected Graph, Components:

A graph G is **connected** if for every $u, v \in V(G)$ there exists a u, v-path in G. Otherwise G is called **disconnected**. The maximal connected subgraphs of G are called its **components**.

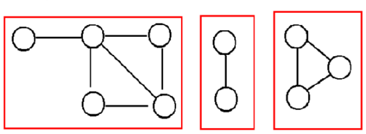
Connected graph:



Disconected graph:

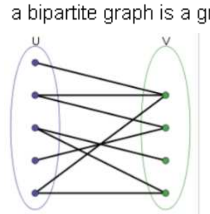


Every disconnected graph can be split up into a number of connected ***components***.
Components:



j) Bipartite graph:

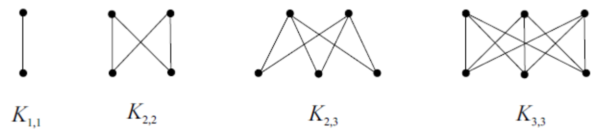
a **bipartite graph** (or **bigraph**) is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V;
a bipartite graph is a graph that does not contain any odd-length cycles.



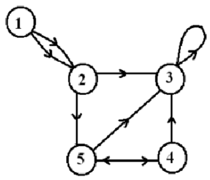
Example of a bipartite graph without cycles

k) Complete Bipartite graph:

A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. The complete bipartite graph with partite sets of size m and n is denoted $K_{m,n}$.



l) Directed graph: a **directed graph** (or **digraph**) is a graph, or set of nodes connected by edges, where the edges have a direction associated with them.



m) In-degree and out-degree, Source and Sink:

For a node, the number of head endpoints adjacent to a node is called the *indegree* of the node and the number of tail endpoints adjacent to a node is its *outdegree* (called "branching factor" in trees).

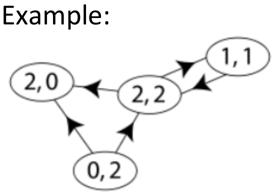
The indegree is denoted $\deg^-(v)$ and the outdegree as $\deg^+(v)$. A vertex with $\deg^-(v) = 0$ is called a *source*, as it is the origin of each of its incident edges. Similarly, a vertex with $\deg^+(v) = 0$ is called a *sink*.

The *degree sum formula* states that, for a directed graph,

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |A| .$$

If for every node $v \in V$, $\deg^+(v) = \deg^-(v)$, the graph is called a *balanced digraph*.[3]

Example:

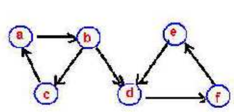


A digraph with vertices labeled (indegree, outdegree)

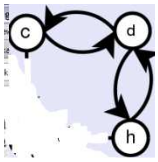
n)Conectivity of Graph: Strongly connected and weakly connected

A digraph G is called *weakly connected* if the undirected *underlying graph* obtained by replacing all directed edges of G with undirected edges is a connected graph. A digraph is *strongly connected* or *strong* if it contains a directed path from *u* to *v* and a directed path from *v* to *u* for every pair of vertices *u*,*v*.

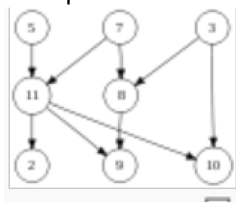
Weakly connected:



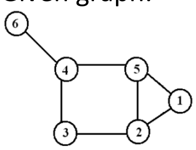
Strongly connected:



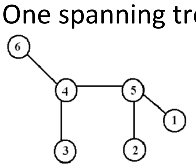
o) DAG: Directed acyclic graph: directed acyclic graph is a directed graph with no directed cycles.
Example:



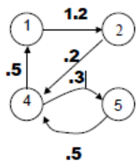
p) Spanning tree and minimum spanning tree:
a **spanning tree** *T* of a connected, undirected graph *G* is a tree composed of all the vertices and some of the edges of *G*.
Given graph:



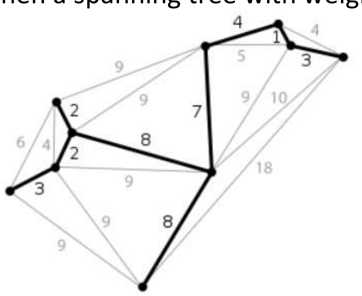
One spanning tree:



q) Weighted graph: A **weighted graph** associates a label (**weight**) with every edge in the graph. Weights are usually real numbers. They may be restricted to rational numbers or integers.

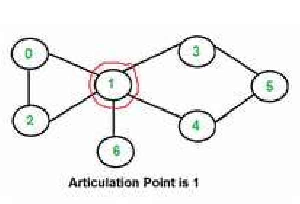


r) minimum spanning tree: A **minimum spanning tree** (**MST**) or **minimum weight spanning tree** is then a spanning tree with weight less than or equal to the weight of every other spanning tree.

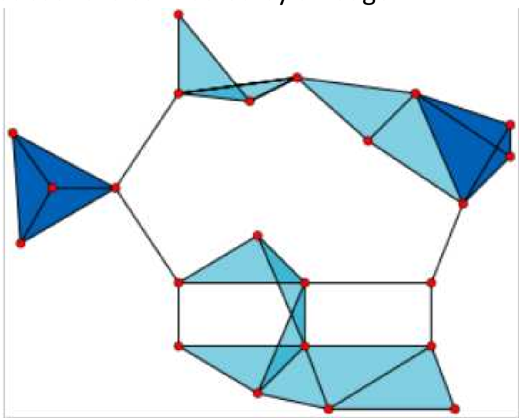


s) Articulation Points (or Cut Vertices):

A vertex in an undirected connected graph is an articulation point (or cut vertex) iff removing it (and edges through it) disconnects the graph.Example:



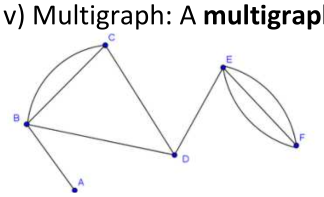
t) Clique: a clique in an undirected graph is a subset of its vertices such that every two vertices in the subset are connected by an edge.



A graph with 23 1-vertex cliques (its vertices), 42 2-vertex cliques (its edges), 19 3-vertex cliques (the light and dark blue triangles), and 2 4-vertex cliques (dark blue areas). The six edges not associated with any triangle and the 11 light blue triangles form maximal cliques. The two dark blue 4-cliques are both maximum and maximal, and the clique number of the graph is 4.

u) Shortest path: Between two vertices u and v the shortest path is the path between them with minimum number of edges.

v) Multigraph: A **multigraph** is a graph that can have more than one edge between a pair of vertices.



1> Some important points:

a)

Proposition 3. *(Degree-Sum Formula) If G is a graph, then*

$$\sum_{v \in V(G)} \deg v = 2 \cdot \#E(G)$$

where $\#E(G)$ is the number of edges in G.

b)

In any graph, the number of vertices of odd degree is even.

c)

A graph G is bipartite if and only if G contains no cycles of odd length.

2> a) Path Matrix:

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \text{if } j\text{th edge lies in the } i\text{th path}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

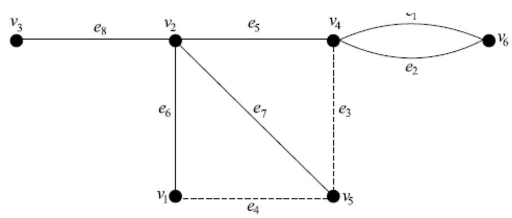


Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$p_1 = \{e_8, e_5\}$, $p_2 = \{e_8, e_7, e_3\}$ and $p_3 = \{e_8, e_6, e_4, e_3\}$.

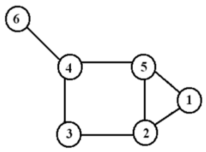
The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

b) incidence matrix:

The **incidence matrix** of G is a $p \times q$ matrix (b_{ij}), where p and q are the numbers of vertices and edges respectively, such that $b_{ij} = 1$ if the vertex v_i and edge x_j are incident and 0 otherwise.

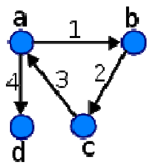
<u>For undirected graph:</u>



	1,2	1,5	2,3	2,5	3,4	4,5	4,6
1	1	1	0	0	0	0	0
2	1	0	1	1	0	0	0
3	0	0	1	0	1	0	0
4	0	0	0	0	1	1	1
5	0	1	0	1	0	1	0
6	0	0	0	0	0	0	1

For directed graph:

The **incidence matrix** of a directed graph D is a $p \times q$ matrix $[b_{ij}]$ where p and q are the number of vertices and edges respectively, such that $b_{ij} = -1$ if the edge x_j leaves vertex v_i, 1 if it enters vertex v_i and 0 otherwise



	1	2	3	4
a	1	0	-1	1
b	-1	1	0	0
c	0	-1	1	0
d	0	0	0	-1

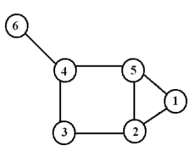
c) adjacency matrix:

An adjacency matrix is a way of representing an n vertex graph $G = (V, E)$ by an $n \times n$ matrix, a, whose entries are boolean values.

The matrix entry $a[i][j]$ is defined as

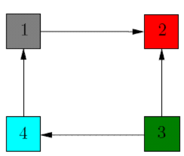
$$a[i][j] = \begin{cases} \text{true} & \text{if } (i, j) \in E \\ \text{false} & \text{otherwise} \end{cases}$$

For undirected graph:



	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	1	0	0
4	0	0	1	0	1	1
5	1	1	0	1	0	0
6	0	0	0	1	0	0

For directed graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$